Systems Programming

Recursion

Julio Villena Román (LECTURER)
<jvillena@it.uc3m.es>

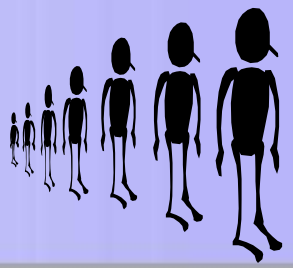
CONTENTS ARE MOSTLY BASED ON THE WORK BY:
Carlos Delgado Kloos

Universidad Carlos III de Madrid

Recursive methods

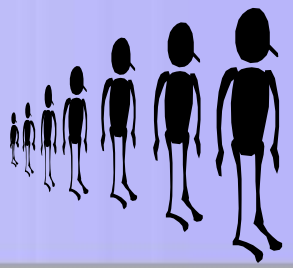
- A method is called recursive, if it calls itself (directly or indirectly)
- (For a recursive method to define a terminating computation)
  the recursive call(s) have to be simpler (according to some metric)

Example 1



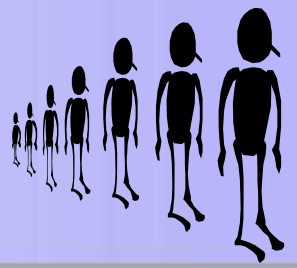
```
public static long s (int n) {
   if (n==1)
      return 1;
   else
      return s(n-1)+n;
}
```

Example 1

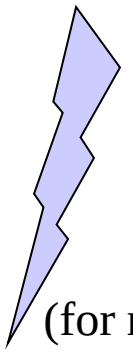


- `s(3) =` (recursive call)
- `s(2)+3 =` (recursive call)
- `(s(1)+2)+3 =` (recursive call)
- `(1+2)+3 =` (sum)
- `(3+3)=` (sum)
- `6`

Example 2

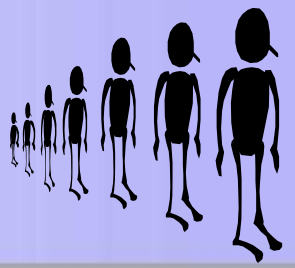


```
public static long s (int n) {
  if (n==1)
     return 1;
  else
     return s(n+1)+n;
}
```



(for n>1)

Example 2

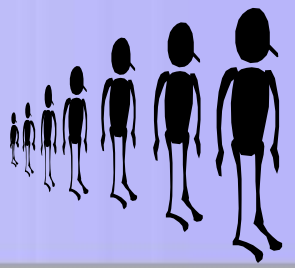


- `s(3) =` (rec. call)
- `s(4)+3 =` (rec. call)
- `(s(5)+4)+3 =` (rec. call)
- `((s(6)+5)+4)+3 =` (rec. call)
- `(((s(7)+6)+5)+4)+3 =` (rec. call)
- `((((s(8)+7)+6)+5)+4)+3 =`
- `...`
- Non-termination

How does a recursive method look like?

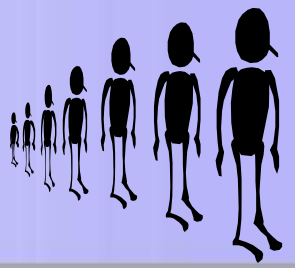
- Conditional
  - Base case (non-recursive) `1`
  - Recursive case `s(n-1)+n`
    (approximates towards the base case condition) `(n==1)`

Exercise: countBack



```
void countBack (int counter) {
  if(counter == 0)
    return;
  else {
    System.out.println(counter);
    countBack(--counter);
    return;
  }
}
```

Exercise: square



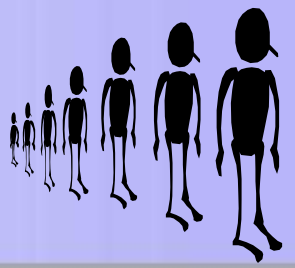
$$(N-1)^2 = N^2 - 2N + 1$$

$$N^2 = (N-1)^2 + 2N - 1$$

$$square(1) = 1$$

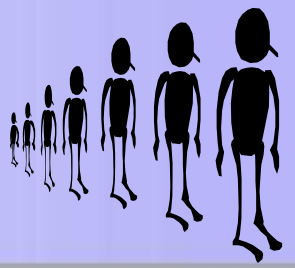
$$square(N) = square(N-1) + 2N - 1$$

Exercise: square



```
int square(int n) {
  if (n == 1)
    return 1;
  else
    return square(n-1)+2*n-1;
}
```

Exercise: mystery



```
mystery(0,Q) = Q
mystery(P,Q) = mystery(P-1, Q+1)
```

- What is the value of mystery(2,4)?

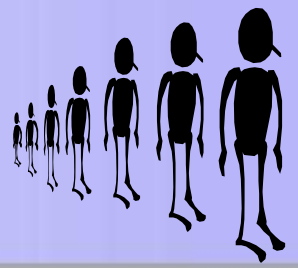
Kinds of recursion: Linear recursion

- Linear recursion
  (in each conditional branch at most one recursive call)
  - **Tail** recursion
    (last operation in branch: recursive call)
  - **Non-tail** recursion
    (pending operation)

Kinds of recursion: Non-linear recursion

- Non-linear recursion
  - Cascading recursion
    `( op(f...,f...) )`
  - Nested recursion
    `( f(...f...) )`
  - ...

Example 3: Factorial



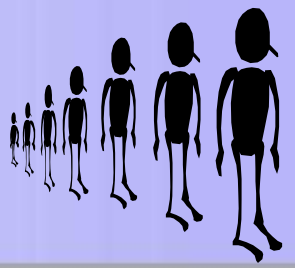
```
fac(n) = n!
```

```
fac(5)=
5*4*3*2*1
```

```
fac(5)=
5*fac(4)=
5*24=120
```

n	fac(n)
0	1
1	1
2	2
3	6
4	24
5	120
...	...

Non-tail recursion: Factorial

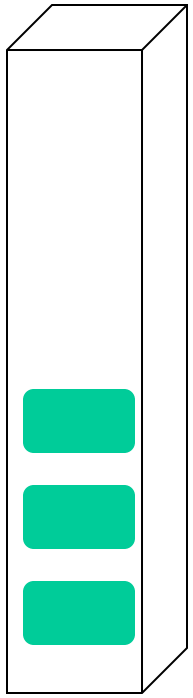


$$\text{fact}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{fact}(n-1) & \text{if } n > 0 \end{cases}$$

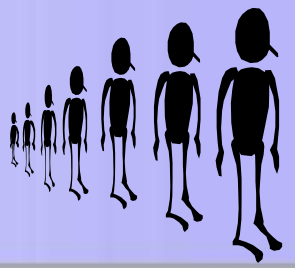
```
public static long fac (int n) {
  if (n<=1)
    return 1;
  else
    return n*fac(n-1);
}
```

Non-tail recursion: Factorial

- `fac(4)`
- `4*fac(3)`
- `4*(3*fac(2))`
- `4*(3*(2*fac(1)))`
- `4*(3*(2*1))`
- `4*(3*2)`
- `4*6`
- `24`

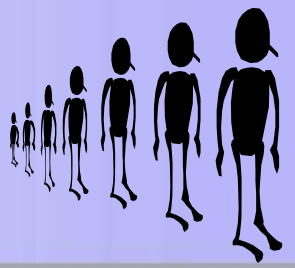


Tail recursion: Factorial



```
public static long fact (int n,m) {
  if (n<=1)
    return m;
  else
    return fact(n-1,n*m);
}

public static long fac (int n) {
  return fact(n,1);
}
```

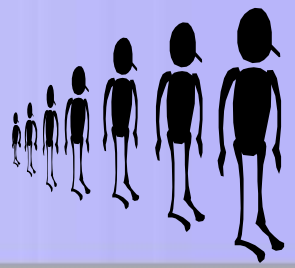
Tail recursion:
Greatest Common Divisor

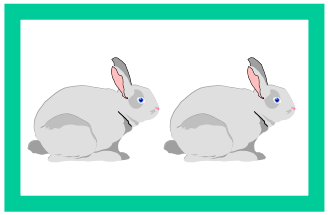


$$\gcd(x,y) = \begin{cases} x & \text{if } y = 0 \\ \gcd(y, \text{remainder}(x,y)) & \text{if } y > 0 \end{cases}$$

```
public static long gcd(int x,y) {
  if (y==0)
    return x;
  else
    return gdc(y, x % y);
}
```

Example 4: Fibonacci

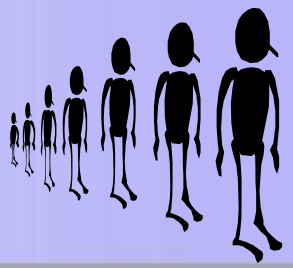




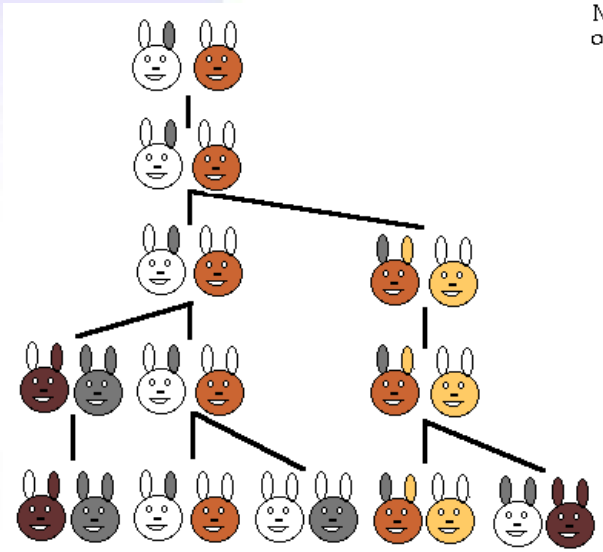
n	fib(n)
0	1
1	1
2	2
3	3
4	5
5	8
...	...

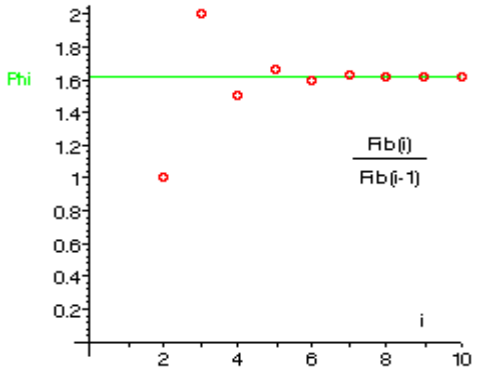
```
fib(5)=
fib(4)+fib(3)
= 5+3
```

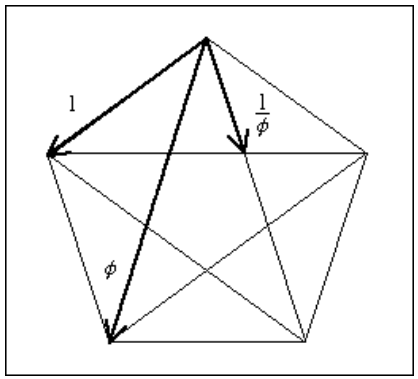
Example 4: Fibonacci

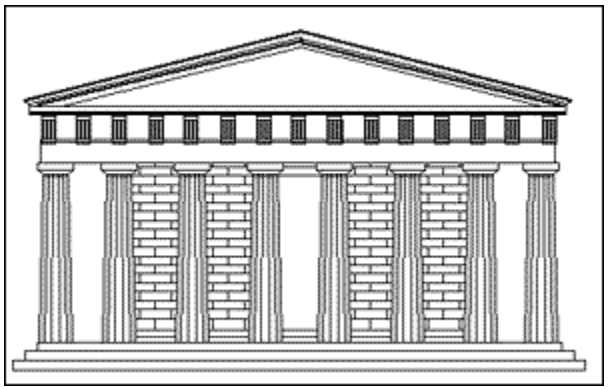


- **Exercise:** Search for applications of *Fibonacci* with *google*

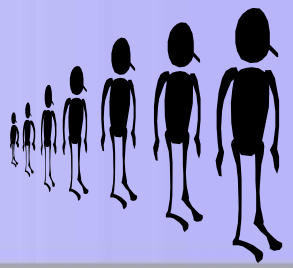






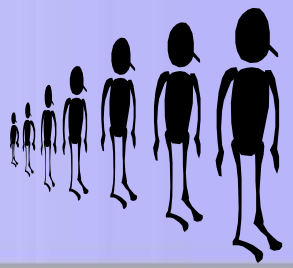


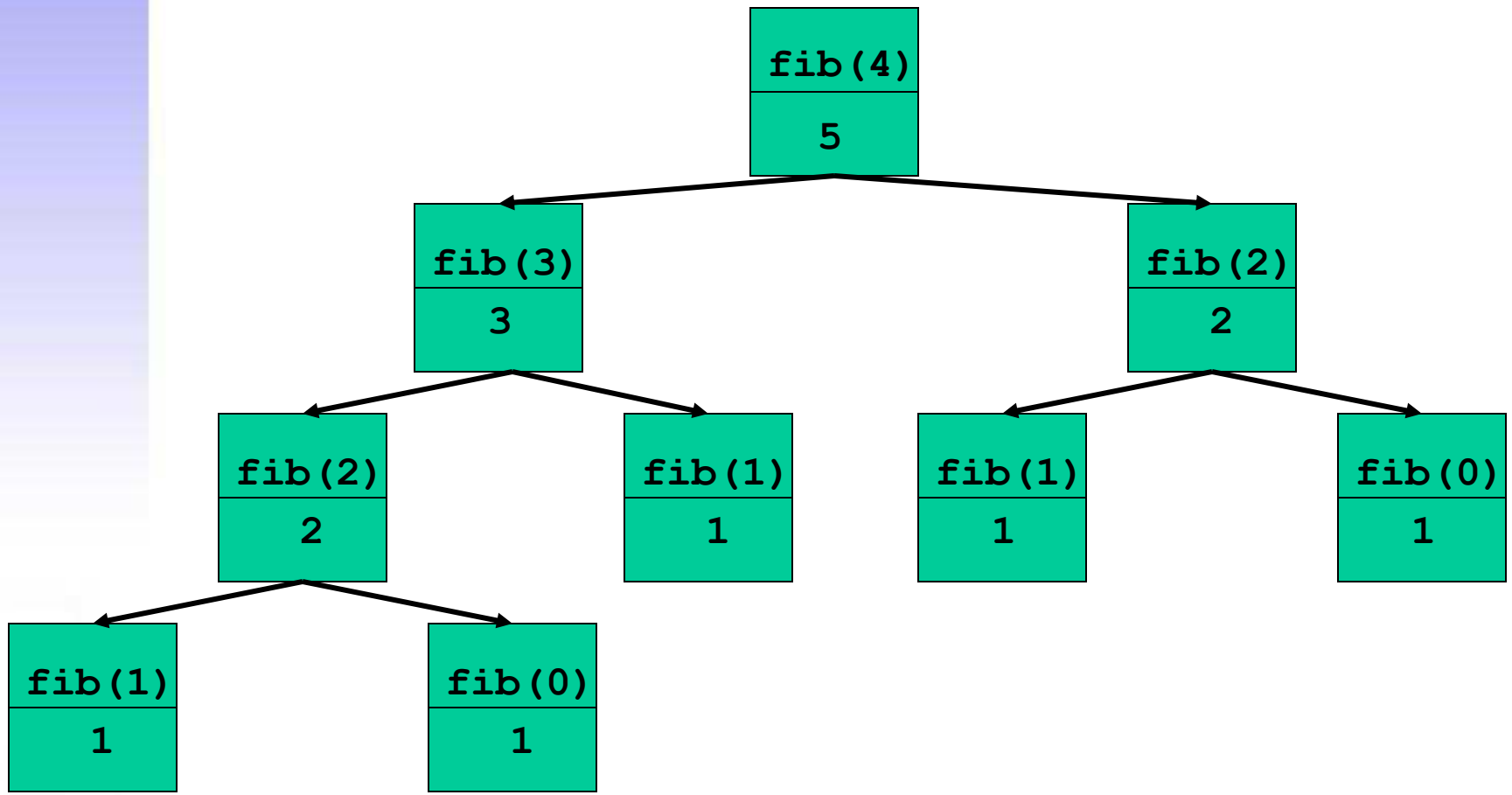
Cascading recursion: Fibonacci



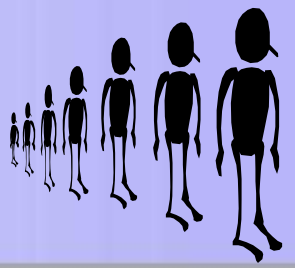
```
public static long fib (int n) {
  if (n<=1)
      return 1;
  else
      return fib(n-1)+fib(n-2);
}
```

Fibonacci



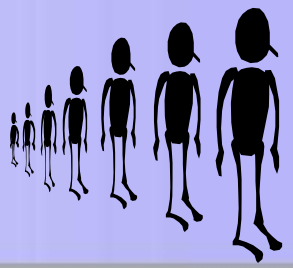


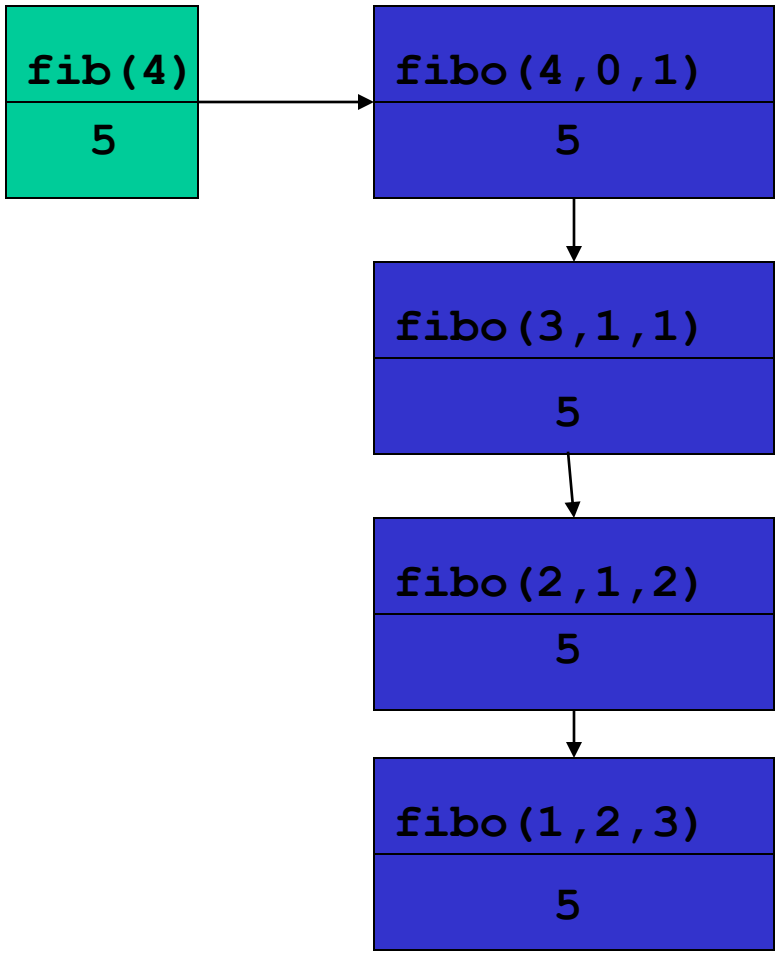
Linear Fibonacci



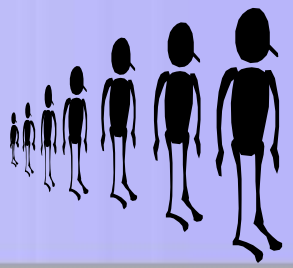
```
public static long fibo(int n,x,y) {
  if (n<=1)
     return x+y;
  else
     return fibo(n-1,y,x+y);
}

public static long fib (int n) {
  return fibo(n,0,1);
}
```

Fibonacci



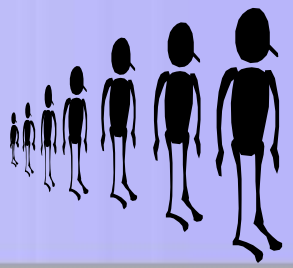


Non-recursive Fibonacci



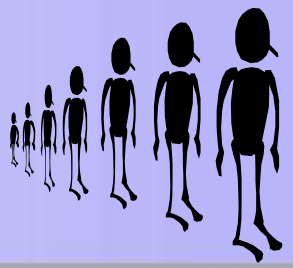
$$fib(n) = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(2^{n+1} \cdot \sqrt{5})}$$

Nested recursion: Morris



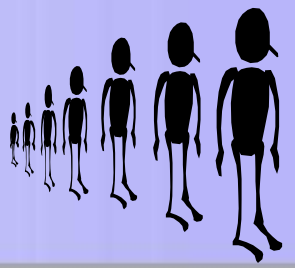
```
public static long mor(int n, m) {
  if (n==m)
     return (m+1);
  else
     return mor(n,mor(n-1,m+1));
}
```

Morris



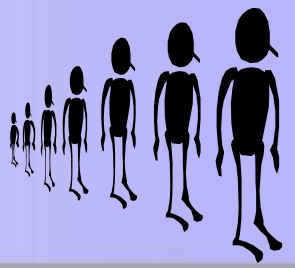
- `mor(4,0)` =
- `mor(4,mor(3,1))` =
- `mor(4,mor(3,mor(2,2))` =
- `mor(4,mor(3,3))` =
- `mor(4,4)` =
- `5`

Nested recursion: Ackermann

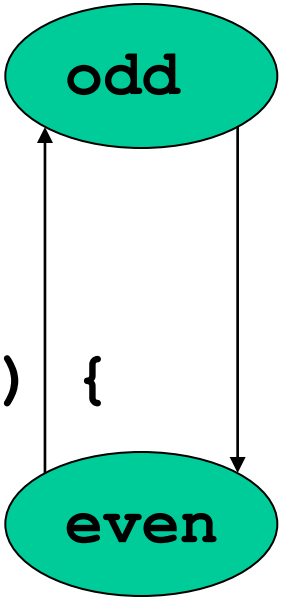


```
public static long ack (int n, m) {
  if (n==0)
     return (m+1);
  else if (m==0)
     return ack(n-1,1);
  else
     return ack(n-1,ack(n,m-1));
}
```

Mutual recursion



```
public static boolean odd (int n) {
 if (n==0)
      return false;
 else
      return even(n-1);
}


public static boolean even(int n) {
 if (n==0)
      return true;
 else
      return odd(n-1);
}
```

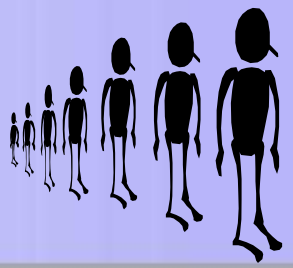


Recursion vs. Iteration

- Tail recursion can be immediately converted into iteration (loop)
- Other forms of recursion require program transformation techniques and possibly more complex data structures

"The transformation from recursion to iteration is one of the most fundamental concepts of computer science." D. Knuth, 1974

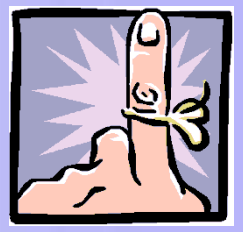
Factorial



```
public long fact (int n,m) {
   if (n<=1)
      return m;
   else
      return fact(n-1,n*m);
}
```

```
public static long fact (int n,m) {
   int M=m, N=n;
   while !(N<=1) {
      N=N-1;
      M=N*M;
   }
   return M;
}
```

To remember



- Range of values for termination
- Base case(s) and decreasing recursive case(s)
- Linear recursion (≤1 recursive call/branch)
  - Tail-recursion (call: last operation, easily convertible to loop) or non-tail recursion
- Non-linear recursion (>1 recursive calls on some branch)
  - Cascading or nested recursion